

Volume computation for sparse boolean quadric relaxations

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Abstract

Motivated by understanding the quality of tractable convex relaxations of intractable polytopes, Ko et al. gave a closed-form expression for the volume of a standard relaxation $\mathcal{Q}(G)$ of the boolean quadric polytope $\mathcal{P}(G)$ of the complete graph $G = K_n$. We extend this work to structured sparse graphs, giving: (i) an efficient algorithm for $\text{vol}(\mathcal{Q}(G))$ when G has bounded tree width, (ii) closed-form expressions (and asymptotic behaviors) for $\text{vol}(\mathcal{Q}(G))$ for all stars, paths, and cycles, and (iii) a closed-form expression for $\text{vol}(\mathcal{P}(G))$ for all cycles. Further, we demonstrate that when G is a cycle, the simple relaxation $\mathcal{Q}(G)$ is a very close model for the much more complicated $\mathcal{P}(G)$. Additionally, we give some computational results demonstrating that this behavior of the cycle seems to extend to more complicated graphs. Finally, we speculate on the possibility of extending some of our results to cactii or even series-parallel graphs.

Keywords: volume, boolean quadric polytope, order polytope, counting linear extensions, bounded tree width, cut polytope

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1. Introduction

For a simple undirected graph $G = (V, E)$ with vertex set $V := \{1, 2, \dots, n\}$ and edge set $E \subset \{(i, j) : 1 \leq i < j \leq n\}$, we let $m := |E|$. The (*graphical*) *boolean quadric polytope* $\mathcal{P}(G)$ is the convex hull in dimension $d := n + \binom{n}{2}$ of the set of binary solutions $\{x_i, y_{ij} : i \in V, i < j \in V\}$ to the system:

$$x_i x_j = y_{ij}, \text{ for each edge } (i, j) \in E.$$

When G is the complete graph K_n , we have the well-known boolean quadric polytope $\mathcal{P}(K_n)$.

The polytope $\mathcal{P}(G)$ is contained in and naturally modelled by $\mathcal{Q}(G)$, the solution set in \mathbb{R}^d of the linear inequalities

$$y_{ij} \leq x_i, \tag{F1}$$

$$y_{ij} \leq x_j, \tag{F2}$$

$$x_i + x_j \leq 1 + y_{ij}, \tag{F3}$$

for $(i, j) \in E$, in which the variables are now relaxed to the continuous interval $[0, 1]$.

Padberg heavily investigated these fundamental polytopes, describing a family of facet-describing inequalities, the affine equivalence of these polytopes with the cut polytope (and its relaxations) of a complete graph, and more; see [30]. Also see [6], [15], [17], [20], [25], [27] and [31] for a glimpse of the literature.

Lee and Morris introduced the idea of using volume as a measure for evaluating relaxations of combinatorial polytopes (see [26], and other work in this vein, e.g., [32] and [36]). In [23], Ko, Lee, and Steingrímsson established that the d -dimensional volume of $\mathcal{Q}(K_n)$ is $2^{2n-d}n!/(2n)!$. At the other end of the spectrum, when G has no edges, $\mathcal{Q}(G) = [0, 1]^d$, the d -dimensional unit hypercube $\{(x, y) : x \in [0, 1]^n, y \in [0, 1]^{\binom{n}{2}}\}$. In what follows, we consider the volume of $\mathcal{Q}(G)$ for cases when $0 < |E(G)| < \binom{n}{2}$. In particular, we focus our attention on sparse G .

In our volume calculations for $\mathcal{Q}(G)$, we are able to consider inequalities F1 and F2 independently from F3. For convenience, we define an additional polytope arising from G . Let $\mathcal{O}(G)$ denote the *order polytope* of G : the subset of $[0, 1]^d$ satisfying inequalities of the form F1 and F2 (but not necessarily F3) for edges $(i, j) \in E$.

We use the notation $\text{vol}_d()$ to denote the d -dimensional volume of a convex body in \mathbb{R}^d . We first observe a simple formula for the volumes of the polytopes that we associate with a graph G , given the volumes of the polytopes associated with the connected components of G .

Lemma 1. *Let $G = (V, E)$ be a simple graph with connected components $G_i = (V_i, E_i)$, $i = 1, 2, \dots, k$. Then*

$$\text{vol}_d(\mathcal{Q}(G)) = \prod_{i=1}^k \text{vol}_{d_i}(\mathcal{Q}(G_i)),$$

where $\mathcal{X} \in \{\mathcal{P}, \mathcal{Q}, \mathcal{O}\}$, $d = |V| + \binom{|V|}{2}$, and $d_i = |V_i| + |E_i|$.

Proof. If $i < j \in V$, but $(i, j) \notin E$, the variable y_{ij} appears in no inequalities defining $\mathcal{X}(G)$ other than $0 \leq y_{ij} \leq 1$. Each of these variables contribute a unit multiplier to the d -dimensional volume calculation of $\mathcal{X}(G)$; in other words, we can ignore them in our volume formulae.

Because the connected components of G share no common vertices or edges, the polytopes $\mathcal{X}(G_i), i \in \{1, 2, \dots, k\}$, are defined on pairwise disjoint sets of variables. Moreover, an inequality of the form F1-F3 involves variables corresponding to exactly one $\mathcal{X}(G_i)$. In other words, there is no interaction between the variables corresponding to different components of G , and the result follows. \square

Corollary 2. *If G is a matching with $m \geq 1$ edges, then $\text{vol}(G) = 1/3^m$.*

Proof. Follows easily from Lemma 1 and Ko et al.'s formula applied to the one-edge graph K_2 . \square

Note that Lemma 1 allows us to mostly restrict our focus to connected graphs. Even when we consider graphs with multiple connected components, we can omit x variables for isolated vertices and y -variables that do not represent edges of G without affecting volumes. In this way, it really does not matter whether we regard our polytopes to be in $\mathbb{R}^{n+\binom{n}{2}}$ or \mathbb{R}^{n+m} , so often we will omit the dimension and just write $\text{vol}(\cdot)$. An exception to this is when we carry out asymptotic analysis, see §6, where the dimension of the ambient space is important.

We conclude this section with a brief overview of the rest of the paper. In §2, we mildly extend the relationship from [23] between $\mathcal{Q}(G)$ and $\mathcal{O}(G)$ for a graph G , and point out how for graphs of bounded tree width (e.g., series-parallel graphs), we can then compute $\text{vol}(\mathcal{Q}(G))$ in polynomial time. In §3, we give a closed-form expression for $\text{vol}(\mathcal{Q}(G))$, when G is a star. In §4, we give a closed-form expression for $\text{vol}(\mathcal{Q}(G))$, when G is a path. Stars and paths are of course forests, and so in those cases, as established by Padberg, we have that $\text{vol}(\mathcal{Q}(G)) = \text{vol}(\mathcal{P}(G))$. In §5, we give closed-form expressions for $\text{vol}(\mathcal{Q}(G))$ and $\text{vol}(\mathcal{P}(G))$, when G is a cycle. We note that our closed-form expressions involve factorial and Euler numbers. For information about calculating them efficiently, we refer the reader to [7], [8], and [19]. In §6, we make asymptotic analyses of the formulae that we have. In particular, we find that $\text{vol}(\mathcal{Q}(G))$ and $\text{vol}(\mathcal{P}(G))$ are quite close when

G is a cycle. In §7, we report on computational experiments designed to see if the behavior of the cycle persists for more complex graphs. In §8, we describe an avenue for further investigation.

2. Order polytopes

The following two results reduce the problem of calculating the volumes of $\mathcal{O}(G)$ and $\mathcal{Q}(G)$ to that of counting the number of linear extensions of a certain poset (partially-ordered set). In particular, let $(\mathcal{V}(G), \prec)$ denote the poset on

$$\mathcal{V}(G) = \{x_i : i \in V\} \cup \{y_{ij} : (i, j) \in E\},$$

with $y_{ij} \prec x_i$ and $y_{ij} \prec x_j$, for all edges $(i, j) \in E$. This poset is known as the *incidence poset* of G . Let $e(\mathcal{V}(G), \prec)$ denote the number of linear extensions of $(\mathcal{V}(G), \prec)$; i.e., the number of order-preserving permutations of $\mathcal{V}(G)$.

Theorem 3. *Let $G = (V, E)$ be a simple graph. Then*

$$\text{vol}(\mathcal{O}(G)) = \frac{e(\mathcal{V}(G), \prec)}{d!},$$

where $d = |V| + |E|$.

Proof. Our polytope $\mathcal{O}(G)$ is an order polytope as described by Stanley in [33]; this result follows directly from Corollary 4.2 in the same paper. \square

Theorem 4. *Let $G = (V, E)$ be a simple graph. Then*

$$\text{vol}(\mathcal{Q}(G)) = \frac{\text{vol}(\mathcal{O}(G))}{2^{|E|}} = \frac{e(\mathcal{V}(G), \prec)}{d!2^{|E|}},$$

where $d = |V| + |E|$.

Proof. We proceed as in the proofs of Proposition 1, Corollary 2, and Proposition 3 in [23]. The primary difference in our case is that we omit variables y_{ij} for which $(i, j) \notin E$.

Define $\mathcal{Q}'(G) := 2\mathcal{Q}(G)$ having facets

$$y_{ij} \leq x_i, \tag{1}$$

$$y_{ij} \leq x_j, \tag{2}$$

$$x_i + x_j \leq 2 + y_{ij}, \tag{3}$$

$$y_{ij} \geq 0, \tag{4}$$

for edges $(i, j) \in E$. It is clear that $\text{vol}(\mathcal{Q}'(G)) = 2^d \text{vol}(\mathcal{Q}(G))$, where $d = |V| + |E|$. Via the same argument as in [23], we partition $\mathcal{Q}'(G)$ into $2^{|V|}$ equi-volume polytopes:

$$R_a := \{(x, y) \in \mathcal{Q}'(G) : a \leq x \leq a + \mathbf{1}\}, a \in \{0, 1\}^{|V|},$$

where $\mathbf{1}$ is the d -vector $(1, 1, \dots, 1)$. In the case of R_0 , the inequality (3) is rendered vacuous, so that R_0 is an order polytope with

$$\text{vol}(R_0) = \frac{e(\mathcal{V}(G), \prec)}{d!}.$$

We conclude that

$$\text{vol}(\mathcal{Q}(G)) = \frac{\text{vol}(\mathcal{Q}'(G))}{2^d} = \frac{2^{|V|} \text{vol}(R_0)}{2^d} = \frac{e(\mathcal{V}(G), \prec)}{2^{|E|} d!}.$$

□

To find the volumes of the relaxation polytopes $\mathcal{O}(G)$ and $\mathcal{Q}(G)$, we must count the number of linear extensions of $(\mathcal{V}(G), \prec)$. In general, counting the number of linear extensions of a poset is #P-complete; see [9]. We are particularly interested in situations when counting the number of linear extensions of $(\mathcal{V}(G), \prec)$ is easier, due to some structured sparsity of G .

For *any* poset (N, \prec) , we can consider its *directed cover graph* $\mathcal{DC}(N, \prec)$, having vertex set N and an edge from vertex i to distinct vertex j when $i \prec j$ and there is no k , distinct from i and j , with $i \prec k \prec j$. We let $\mathcal{C}(N, \prec)$ denote the *cover graph*, ignoring the edge directions in $\mathcal{DC}(N, \prec)$.

Cover graphs of incidence posets have been studied (see [37], for example). We are interested in sparsity properties of $\mathcal{C}(G) := \mathcal{C}(\mathcal{V}(G), \prec)$, the cover graph of the incidence poset of G . See Figure 1 for an example of a simple graph G and the related directed cover graph $\mathcal{DC}(G)$.

It is clear that $\mathcal{C}(G)$ inherits all of the graph properties of G that are inherited by edge subdivision. For example, if G is a tree, cycle, cactus, or series-parallel graph, then $\mathcal{C}(G)$ is a tree, cycle, cactus, or series-parallel graph, respectively. Furthermore, if G has tree-width bounded by k , then $\mathcal{C}(G)$ has tree-width bounded by k : whenever G has tree-width k , then inserting a vertex on each edge leaves the tree-width at k (this is mentioned and used in many papers, but see [28, Lemma 1] where it is explicitly stated and proved). Besides trees ($k = 1$), cycles ($k = 2$), cactii ($k = 2$), and

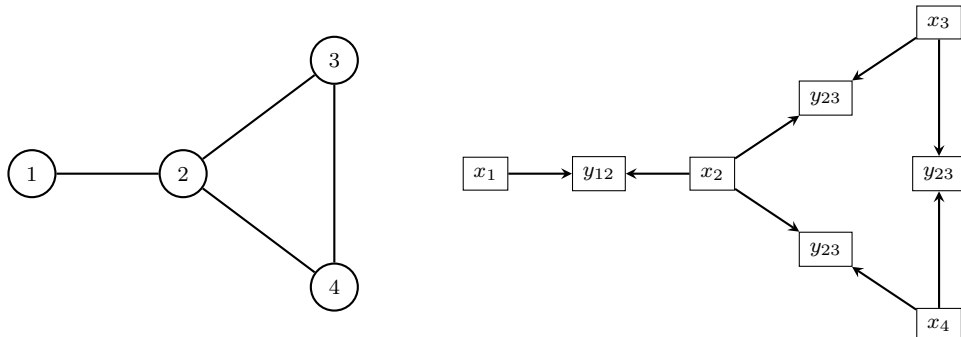


Figure 1: G and $\mathcal{DC}(G)$

series-parallel graphs ($k = 2$), we can consider outer-planar graphs ($k = 2$) and Halin graphs ($k = 3$).

Structured sparsity of $\mathcal{C}(G)$ can be exploited. [2] and [3] give polynomial-time algorithms for counting the number of linear extensions when $\mathcal{C}(N, \prec)$ is a tree. [10] extended this to the case in which $\mathcal{C}(N, \prec)$ is a cactus (also see[2]). [18] extended this to the case in which $\mathcal{C}(N, \prec)$ has bounded tree-width. We have then the following fundamental result.

Theorem 5. *For the class of graphs of tree-width bounded by any constant, in polynomial time, we can calculate $\text{vol}(\mathcal{O}(G))$ and $\text{vol}(\mathcal{Q}(G))$.*

Interestingly, [9] proved that counting the number of linear extensions of a poset is #P-complete even for posets of height 3, and they conjectured that this is true even for posets of height 2. The height-2 situation is very relevant to our investigation because our posets $(\mathcal{V}(G), \prec)$ have height 2. However, our posets are rather special posets of height 2, in that all of our y vertices have degree 2, so a positive complexity result is more likely. In any case, Brightwell and Winkler asserted¹ that: (i) the complexity for the general height-2 case is still open; (ii) there seems to be no work on counting linear extensions of incidence posets; (iii) there is no compelling reason to believe that the case of incidence posets should be easier than general height-2 posets.

¹January 16, 2017, private communication.

3. Stars

Let S_m denote a star with $m \geq 1$ edges.

Lemma 6. *For $m \geq 1$,*

$$e(\mathcal{V}(S_m), \prec) = 2^m(m!)^2.$$

Proof. Let the vertex set of the star be $V := \{0, 1, \dots, m\}$, and let vertex 0 be the center of the star. For convenience, we count the reverse linear extensions of $e(\mathcal{V}(S_m), \prec)$, in which x_0 and x_k appear to the left of $y_{0,k}$.

For $i = 0, 1, \dots, m$, the number of permissible permutations of $\mathcal{V}(S_m)$ in which x_0 appears in position $i + 1$ and all other x variables are ordered by index is given by

$$\left(i! \binom{2m-i}{i} \right) \left(\frac{(2(m-i))!}{2^{m-i}(m-i)!} \right).$$

The first factor counts the number of ways to place $y_{0,1}, y_{0,2}, \dots, y_{0,i}$ into the $2m - i$ positions to the right of x_0 in no particular order. The second factor counts the placement of the pairs $x_k, y_{0,k}$, for $i + 1 \leq k \leq m$, in the remaining $2(m - i)$ positions to the right of x_0 with $x_k \prec y_{0,k}$ and the x_k ordered by index.

Incorporating all possible positions of x_0 and permutations of the $x_k, y_{0,k}$ pairs for $1 \leq k \leq m$, we obtain

$$\begin{aligned} e(\mathcal{V}(S_m), \prec) &= m! \sum_{i=0}^m i! \binom{2m-i}{i} \left(\frac{(2(m-i))!}{2^{m-i}(m-i)!} \right) \\ &= m! \sum_{i=0}^m \frac{(2m-i)!}{(m-i)! 2^{m-i}} \\ &= 2^m(m!)^2 \sum_{i=0}^m \binom{2m-i}{m-i} \frac{1}{2^{2m-i}} \\ &= 2^m(m!)^2 \sum_{\ell=0}^m \binom{m+\ell}{\ell} \frac{1}{2^{m+\ell}}. \end{aligned}$$

To see that $\sum_{\ell=0}^m \binom{m+\ell}{\ell} \frac{1}{2^{m+\ell}} = 1$, rewrite the summation as

$$\sum_{\ell=0}^m \binom{m+\ell}{\ell} \left[\left(\frac{1}{2} \right)^{m+1} \left(\frac{1}{2} \right)^{\ell} + \left(\frac{1}{2} \right)^{m+1} \left(\frac{1}{2} \right)^{\ell} \right],$$

which represents the probability that a fair coin will land on the same side (heads or tails) exactly $m+1$ times somewhere between flips $m+1$ and $2m+1$. This identity is a special case (in which $x = \frac{1}{2}$) of an identity apparently due to Gosper (see [5, Item 42]). \square

Combining Lemma 6 and Theorem 4, we obtain the following result.

Theorem 7. *For $m \geq 1$,*

$$\text{vol}(\mathcal{Q}(S_m)) = \frac{\text{vol}(\mathcal{O}(S_m))}{2^m} = \frac{(m!)^2}{(2m+1)!}.$$

4. Paths

We will see that the so-called odd “Euler numbers” appear in the formulae for the volumes of polytopes associated with paths (and cycles). There are several closely related sequences that are called “Euler numbers”, so there can be considerable confusion. Following [35], an *alternating permutation* of $\{1, 2, \dots, k\}$ is a permutation so that each entry is alternately greater or less than the preceding entry. The *Euler number* A_k , $k \geq 1$, is the number of such alternating permutations, and *André’s problem* is determining the A_k . For small values, we have

$$A_0 := 1, A_1 = 1, A_2 = 1, A_3 = 2, A_4 = 5, A_5 = 16, A_6 = 61, A_7 = 272.$$

Even-indexed Euler numbers are also called *zig numbers*, and the odd-indexed ones are called *zag numbers*. Also, the even-indexed ones are called *secant numbers*, and the odd-indexed ones are called *tangent numbers*. The latter names come from André who established the following pretty result.

Theorem 8 (D. André, [1]). *The Maclaurin series for $\sec(x)$ is $\sum_{m=0}^{\infty} \frac{A_{2m}}{(2m)!} x^m$, and the Maclaurin series for $\tan(x)$ is $\sum_{m=0}^{\infty} \frac{A_{2m+1}}{(2m+1)!} x^m$, both having radius of convergence $\pi/2$.*

Note that Euler worked with the odd-indexed ones, and defined them not combinatorially but rather via the Maclaurin series for $\tan(x)$ (see [35]).

Let P_m denote a path with $m \geq 0$ edges.

Lemma 9. *For $m \geq 0$,*

$$e(\mathcal{V}(P_m), \prec) = A_{2m+1}.$$

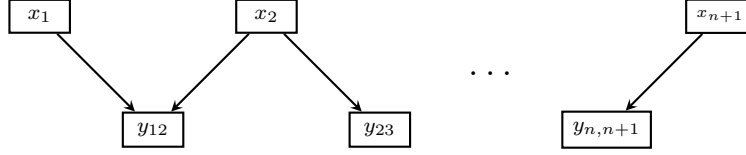


Figure 2: $\mathcal{C}(P_m)$

Proof. Figure 2 is a diagram of $\mathcal{C}(P_m)$. The number $e(\mathcal{V}(P_m), \prec)$ counts the number of permutations ϕ of

$$\{x_1, y_{1,2}, x_2, y_{2,3}, x_3, \dots, x_m, y_{m,m+1}, x_{m+1}\},$$

such that

$$\phi(x_1) > \phi(y_{1,2}) < \phi(x_2) > \phi(y_{2,3}) < \dots > \phi(y_{m,m+1}) < \phi(x_{m+1}).$$

In other words, $e(\mathcal{V}(P_m), \prec)$ is precisely the number of alternating permutations of a $(2m+1)$ -element set, which is given by the odd Euler number, A_{2m+1} . \square

Combining Lemma 9 and Theorems 3 and 4, we obtain the following result.

Theorem 10. *For $m \geq 0$,*

$$\text{vol}(\mathcal{Q}(P_m)) = \frac{\text{vol}(\mathcal{O}(P_m))}{2^m} = \frac{A_{2m+1}}{2^m(2m+1)!}.$$

Because of André's Theorem, we can see the volumes of the boolean quadric polytopes corresponding to paths in the McLaurin series expansion of tangent.

Corollary 11.

$$\sum_{m \geq 0} 2^m \text{vol}(\mathcal{Q}(P_m)) x^{2m+1} = \tan(x).$$

5. Cycles

In this section we obtain a first success at fully analyzing a situation where $\mathcal{Q}(G)$ is different from $\mathcal{P}(G)$. Let C_m be a simple cycle with $m \geq 3$ edges.

Theorem 12 (Kreweras, [24]). *For $m \geq 3$,*

$$e(\mathcal{V}(C_m), \prec) = mA_{2m-1}.$$

Proof. The number of cyclically alternating permutations of length $2m$ is precisely mA_{2m-1} ; see [24]. \square

Combining Theorem 12 and Theorem 4, we obtain the following result.

Theorem 13. *For $m \geq 3$,*

$$\text{vol}(\mathcal{Q}(C_m)) = \frac{\text{vol}(\mathcal{O}(C_m))}{2^m} = \frac{mA_{2m-1}}{2^m(2m)!}.$$

Combining Theorems 13 and 10, we obtain the following result.

Corollary 14. *For $m \geq 3$,*

$$\text{vol}(\mathcal{Q}(C_m)) = \text{vol}(\mathcal{Q}(P_{m-1}))/4.$$

Padberg made a careful analysis of the facets of $\mathcal{P}(G)$ when G is a cycle (see [30]). We summarize the relevant parts in the remainder of this paragraph. Let $G = (V, E)$ be a simple graph containing simple cycle $C = (V(C), E(C))$. Let A be an odd cardinality subset of $E(C)$ and define

$$\begin{aligned} S_0 &:= \{i \in V(C) : i \text{ is incident to no elements of } A\}; \\ S_1 &:= \{i \in V : i \notin V(C) \text{ or } i \text{ is incident to exactly 1 element of } A\}; \\ S_2 &:= \{i \in V(C) : i \text{ is incident to 2 elements of } A\}. \end{aligned}$$

Note that $V = S_0 \cup S_1 \cup S_2$. The *odd cycle inequality* $OC(A)$,

$$\sum_{i \in S_2} x_i - \sum_{i \in S_0} x_i + \sum_{(i,j) \in E \setminus A} y_{ij} - \sum_{(i,j) \in A} y_{ij} \leq \lfloor |A|/2 \rfloor, \quad (\text{F4})$$

is a valid inequality of $\mathcal{P}(G)$ and cuts off the vertex of $\mathcal{Q}(G)$ given by

$$\begin{aligned} x_i &= \frac{1}{2}, \text{ for } i \in V, \\ y_{ij} &= \frac{1}{2}, \text{ for } (i, j) \in E \setminus A, \text{ and} \\ y_{ij} &= 0, \text{ for } (i, j) \in A \cup (E \setminus E(C)). \end{aligned}$$

When G is a cycle, the inequalities F4 are facets of $\mathcal{P}(G)$ and, moreover, $\mathcal{P}(G)$ is completely described by F1-F4. In fact, $\mathcal{P}(G)$ is completely described by F1-F4 for any series-parallel graph G .

Next, we look carefully at the parts of $\mathcal{Q}(G)$ cut off by distinct odd cycle inequalities arising from a single simple cycle C of G . We will see that they are disjoint, and for the special case that $G = C$, the cut off parts all have the same volume which we can calculate. Note that this is a similar behavior to so-called “clipping inequalities” applied to the standard unit hypercube (see [11]).

Lemma 15. *Let G be a simple graph containing simple cycle C . Let A and B be distinct odd-cardinality subsets of $E(C)$. The odd cycle inequalities $OC(A)$ and $OC(B)$ remove disjoint portions of $\mathcal{Q}(G)$.*

Proof. Let A and B be distinct odd-cardinality subsets of $E(C)$. Let S_0, S_1 , and S_2 be as defined above for A . Let T_0, T_1 , and T_2 be the corresponding subsets of V related to B . Points removed by both $OC(A)$ and $OC(B)$ must satisfy the inequalities

$$\sum_{i \in S_2} x_i - \sum_{i \in S_0} x_i + \sum_{(i,j) \in E \setminus A} y_{ij} - \sum_{(i,j) \in A} y_{ij} > \lfloor |A|/2 \rfloor$$

and

$$\sum_{i \in T_2} x_i - \sum_{i \in T_0} x_i + \sum_{(i,j) \in E \setminus B} y_{ij} - \sum_{(i,j) \in B} y_{ij} > \lfloor |B|/2 \rfloor.$$

Adding these inequalities, canceling terms with opposite signs, we have

$$\begin{aligned} & \sum_{i \in S_2 \cap T_1} x_i + \sum_{i \in S_1 \cap T_2} x_i + 2 \sum_{i \in S_2 \cap T_2} x_i - 2 \sum_{(i,j) \in A \cap B} y_{ij} \\ & - \sum_{i \in S_0 \cap T_1} x_i - \sum_{i \in S_1 \cap T_0} x_i - 2 \sum_{i \in S_0 \cap T_0} x_i + 2 \sum_{(i,j) \in E \setminus (A \cup B)} y_{ij} \\ & > \frac{|A| + |B|}{2} - 1. \quad (5) \end{aligned}$$

We are only interested in points that are in $\mathcal{Q}(G)$, so we apply $F1 - F3$ and non-negativity to (5). Add $y_{ij} - x_i - x_j \geq -1$ for $(i, j) \in A \cap B$ and $x_i + x_j - 2y_{ij} \geq 0$ for $(i, j) \in E(C) \setminus (A \cup B)$ to obtain

$$\sum_{i \in W} x_i - \sum_{(i,j) \in A \cap B} y_{ij} - \sum_{i \in W} x_i > \frac{|A| + |B|}{2} - |A \cap B|, \quad (6)$$

where $W \subseteq S_1 \cap T_1$ is the set of vertices adjacent to $A \cap B$ (and to $E(C) \setminus (A \cup B)$) that are not already covered by $(S_2 \cap T_1) \cup (S_1 \cap T_2) \cup (S_2 \cap T_2)$ (nor by $(S_0 \cap T_1) \cup (S_1 \cap T_0) \cup (S_0 \cap T_0)$). Apply non-negativity constraints to eliminate the remaining y terms, arriving at

$$0 > \frac{|A| + |B|}{2} - |A \cap B| \geq \min\{|A|, |B|\} - |A \cap B| \geq 0, \quad (7)$$

a contradiction. \square

Lemma 16. *The volume removed from $\mathcal{Q}(C_m)$ by a single odd-cycle constraint is $\frac{1}{2(2m)!}$.*

Proof. Suppose that A is any odd-cardinality subset of the edges of C_m , $E = \{(1, 2), (2, 3), \dots, (2m-1, 2m), (2m, 1)\}$. According to [30], the portion of $\mathcal{Q}(C_m)$ cut off by $OC(A)$ is a simplex, which we denote W . If v^0, v^1, \dots, v^{2m} are the vertices of W ,

$$\text{vol}(W) = \frac{1}{(2m)!} \left| \det(v^1 - v^0, v^2 - v^0, \dots, v^{2m} - v^0) \right|.$$

Our task is to show that the determinant in the formula above is $\pm 1/2$.

Let v^0 be the fractional vertex of $\mathcal{Q}(C_m)$ that violates $OC(A)$. We have already seen that all coordinates of v^0 are $1/2$ except those corresponding to edges in A , which have value 0. There are $2m$ facets of $\mathcal{Q}(C_m)$ that meet at v^0 . There is a pair of these facets for each edge (i, j) , and the form of this pair of facets depends on whether or not (i, j) is in A . If edge (i, j) is in A , v^0 satisfies $y_{ij} \geq 0$ and $x_i + x_j \leq y_{ij} + 1$ with equality. If (i, j) is not in A , v^0 satisfies $y_{ij} \leq x_i$ and $y_{ij} \leq x_j$ with equality.

Each of the vertices v^1, v^2, \dots, v^{2m} is integer-valued and arises from the relaxation of one (and only one) of the facets of $\mathcal{Q}(C_m)$ that meet at v^0 . There are four types of these vertices, one for each type of facet of $\mathcal{Q}(C_m)$. The relaxed facet (along with its tight partner) determines the variables corresponding to the related edge and its endpoints as follows:

(i, j)	relaxed facet	x_i	y_{ij}	x_j
A	$y_{ij} \geq 0$	1	1	1
A	$x_i + x_j \leq y_{ij} + 1$	0	0	0
\bar{A}	$x_i \geq y_{ij}$	1	0	0
\bar{A}	$x_j \geq y_{ij}$	0	0	1

The rest of the coordinates of v^k are completely determined by the other $2m - 2$ constraints that are tight at v^0 (which are also tight at v^k). If (i, j) is an edge that is not associated with the relaxed facet for v^k , the possible values of x_i , y_{ij} , and x_j are

(i, j)	x_i	y_{ij}	x_j
A	1	0	0
A	0	0	1
\bar{A}	1	1	1
\bar{A}	0	0	0

because if $(i, j) \in A$, we have $y_{ij} = 0$ and $x_i + x_j = y_{ij} + 1$, and if $(i, j) \in \bar{A}$ we have $x_i = y_{ij} = x_j$. In other words if we assign values of the variables in the order that the associated vertices and edges occur in C_m , starting with the coordinates that are determined by the relaxed facet, if $(i, j) \in A$ then the variable values switch (0 to 1 or 1 to 0) from x_i to x_j , whereas if $(i, j) \in \bar{A}$, the values are unchanged from x_i to x_j .

As we set up our matrix, $(v^1, v^2, \dots, v^{2m})$, we have a choice in the order of the vertices. We make the following assignments for $i = 1, 2, \dots, m$, depending again on whether the associated edge is in A . If $(i, j) \in A$, we assign

vertex	relaxed facet
v^{2i-1}	$y_{ij} \geq 0$
v^{2i}	$x_i + x_j \leq y_{ij} + 1$

If $(i, j) \in \bar{A}$, then assign

vertex	relaxed facet
v^{2i-1}	$x_i \geq y_{ij}$
v^{2i}	$x_j \geq y_{ij}$

We claim that for $k = 2, \dots, 2m$, the coordinates of $\mathbf{v}^{k+} := (\mathbf{v}^k - \mathbf{v}^0) + (\mathbf{v}^{k-1} - \mathbf{v}^0) = \mathbf{v}^k + \mathbf{v}^{k-1} - 2\mathbf{v}^0$ or $\mathbf{v}^{k-} := (\mathbf{v}^k - \mathbf{v}^0) - (\mathbf{v}^{k-1} - \mathbf{v}^0) = \mathbf{v}^k - \mathbf{v}^{k-1}$ are all 0 except for the k th and possibly the $(k+1)$ st, and the non-zero coordinates are ± 1 . There are six cases to check. Note that if the row index k is even, both involved vertices correspond to the relaxation of a facet involving the same edge, so three variables (corresponding to the edge and its endpoints) are involved in the facet relaxations. If k is odd, the facets involve two adjacent edges, and so five variables are involved in the two facet relaxations.

Case 1. $k := 2i$ is even and the facets relaxed to form \mathbf{v}^k and \mathbf{v}^{k-1} both correspond to $(i, j) \in A$. As noted above, we have the following assignments of the variables involved in the facet that is relaxed to form each vertex:

	x_i	y_{ij}	x_j
\mathbf{v}^0	1/2	0	1/2
\mathbf{v}^{k-1}	1	1	1
\mathbf{v}^k	0	0	0

Note that via the patterns forced by the facets that are tight at these vertices, the remaining variables maintain the opposite 0/1 pattern initiated at x_j throughout the rest of x_V and all of $y_{\bar{A}}$ in \mathbf{v}^k and \mathbf{v}^{k-1} , whereas $y_A = \mathbf{0}$ in all three vertices listed above. We conclude that $\mathbf{v}^{k+} = \mathbf{e}_k$.

Case 2. $k := 2i$ is even and the facets relaxed to form \mathbf{v}^k and \mathbf{v}^{k-1} both correspond to $(i, j) \in \bar{A}$. We have the assignments

	x_i	y_{ij}	x_j
\mathbf{v}^0	1/2	1/2	1/2
\mathbf{v}^{k-1}	1	0	0
\mathbf{v}^k	0	0	1

and the pattern of opposite values initiated by x_j continues to x_V and the remainder of $y_{\bar{A}}$. Again $y_{\bar{A}} = \mathbf{0}$ for all three vertices, and $\mathbf{v}^{k+} = -\mathbf{e}_k$.

Case 3. $k := 2i - 1$ is odd; $\{(i - 1, i), (i, j)\} \subseteq A$. Now five variables are affected by the the relaxed facets and we have

	x_{i-1}	$y_{i-1,i}$	x_i	y_{ij}	x_j
v^{k-1}	0	0	0	0	1
v^k	0	0	1	1	1

The matching pattern begun at x_j in v^{k-1} and v^k continues through the rest of the x_V and $y_{\bar{A}}$, and $v^{k-} = \mathbf{e}_k + \mathbf{e}_{k+1}$.

Case 4. $k := 2i - 1$ is odd; $\{(i - 1, i), (i, j)\} \subseteq \bar{A}$. We have

	x_{i-1}	$y_{i-1,i}$	x_i	y_{ij}	x_j
v^0	1/2	1/2	1/2	1/2	1/2
v^{k-1}	0	0	1	1	1
v^k	1	1	1	0	0

and $v^{k+} = \mathbf{e}_k$.

Case 5. $k := 2i - 1$ is odd; $(i - 1, i) \in A$ and $(i, j) \in \bar{A}$. In this case, assignments are

	x_{i-1}	$y_{i-1,i}$	x_i	y_{ij}	x_j
v^{k-1}	0	0	0	0	0
v^k	0	0	1	0	0

and $v^{k-} = \mathbf{e}_k$.

Case 6. $k := 2i - 1$ is odd; $(i - 1, i) \in \bar{A}$ and $(i, j) \in A$. In this last case, we have

	x_{i-1}	$y_{i-1,i}$	x_i	y_{ij}	x_j
v^0	1/2	1/2	1/2	0	1/2
v^{k-1}	0	0	1	0	0
v^k	1	1	1	1	1

for which $v^{k+} = e_k + e_{k+1}$.

Via a sequence of row operations that leaves the determinant unchanged, we can transform $(v^1 - v^0, v^2 - v^0, \dots, v^{2m} - v^0)$ into the upper triangular matrix $M := (v^1 - v^0, v^{2+}, v^{3\pm}, v^{4+}, v^{5\pm}, \dots, v^{2m+})$, where the rows indexed by $3, 5, \dots, 2m-1$ are chosen as in cases 3-6 above. Whether or not edge $(1, 2)$ is in A , $M_{1,1} = 1/2$ and, as shown above, $M_{k,k} = \pm 1$ for $2 \leq k \leq 2m$. \square

Combining Theorem 13 with Lemmas 15 and 16, we obtain the following result — a closed-form expression for the volume of the boolean quadric polytope of a cycle.

Theorem 17. *For $m \geq 3$,*

$$\text{vol}(\mathcal{P}(C_m)) = \text{vol}(\mathcal{Q}(C_m)) - \frac{2^{m-1}}{2(2m)!} = \frac{mA_{2m-1}}{2^m(2m)!} - \frac{2^{m-2}}{(2m)!}.$$

6. Asymptotics

As indicated in [26], it is natural to compare sets in \mathbb{R}^d by comparing the d -th roots of their volumes. Because we take d -th roots, we have to be precise about the ambient dimension d for our polytopes. So in what follows we assume that our graphs have no isolated vertices, and we always regard our polytopes as being in dimension $d = n + m$, the least dimension that makes sense (rather than in dimension $n + \binom{n}{2}$).

As we have mentioned at the outset, [23] established the d -dimensional volume of $\mathcal{Q}(K_n)$ to be $2^{2n-d}n!/(2n)!$, where $d = n + \binom{n}{2}$. Invoking Stirling's formula, it is easy to check the following calculation.

Proposition 18.

$$\lim_{n \rightarrow \infty} \text{vol}_d(\mathcal{Q}(K_n))^{1/d} = \frac{1}{2}, \quad (8)$$

where $d = n + \binom{n}{2}$.

This is quite substantial, as the volume of the entire unit hypercube and its d -th root are of course unity. It is an outstanding open problem, first posed in [23] and which we would like to highlight, to understand how close $\text{vol}_d(\mathcal{Q}(K_n))$ and $\text{vol}_d(\mathcal{P}(K_n))$ are, asymptotically.

When G is a forest \mathcal{P} and \mathcal{Q} are the same. Still it is interesting to compare the asymptotics of $\mathcal{Q}(G)$ and $\mathcal{Q}(G')$ when connected G and G' have the same number of edges. Our next result does this for two very different trees on m edges.

Corollary 19.

$$\lim_{m \rightarrow \infty} \text{vol}_{2m+1}(\mathcal{Q}(S_m))^{1/(2m+1)} = \frac{1}{2} \quad (9)$$

and

$$\lim_{m \rightarrow \infty} \text{vol}_{2m+1}(\mathcal{Q}(P_m))^{1/(2m+1)} = \frac{\sqrt{2}}{\pi} \approx 0.450158. \quad (10)$$

Proof. It is easy to check (9) using Stirling's formula.

By Theorem 13, we have that $\text{vol}_{2m}(\mathcal{Q}(C_m)) = \frac{mA_{2m-1}}{2^m(2m)!}$. By André's Theorem 8, we have

$$A_k/k! = \frac{4}{\pi} \left(\frac{2}{\pi}\right)^k + \mathcal{O}\left(\left(\frac{2}{3\pi}\right)^k\right)$$

(see [34]). Combining these facts, (10) follows easily. \square

It is interesting to observe that the path and star, both on m edges and $m+1$ vertices, and hence having their associated polytopes naturally living in dimension $2m+1$, behave substantially similarly though non-trivially differently, from our point of view.

Next, we demonstrate that for C_m , the volume of \mathcal{Q} is quite large compared to the volume of \mathcal{Q} that is outside of \mathcal{P} . So, we have a family of examples demonstrating that \mathcal{P} can have a description that has many more inequalities than \mathcal{Q} , while their volumes are very close. In particular, when G is a cycle, \mathcal{Q} has only $4m$ facets, while \mathcal{P} has $4m + 2^{m-1}$ facets.

Corollary 20.

$$\lim_{m \rightarrow \infty} \text{vol}_{2m}(\mathcal{Q}(C_m))^{1/2m} = \frac{\sqrt{2}}{\pi} \approx 0.450158 \quad (11)$$

and

$$\lim_{m \rightarrow \infty} m \times \text{vol}_{2m}(\mathcal{Q}(C_m) \setminus \mathcal{P}(C_m))^{1/2m} = \frac{e}{\sqrt{2}}. \quad (12)$$

Proof. By Corollary 13, we have that $\text{vol}_{2m}(\mathcal{Q}(C_m)) = \frac{A_{2m-1}}{2^{m+1}(2m-1)!}$. Using again André's Theorem and Stirling's formula (as in the proof of Corollary 19), (11) follows easily.

Because $\mathcal{P}(C_m) \subseteq \mathcal{Q}(C_m)$, Corollary (13) implies that

$$\text{vol}_{2m}(\mathcal{Q}(C_m) \setminus \mathcal{P}(C_m)) = 2^{m-2}/(2m)!.$$

Invoking Stirling's formula, we have

$$\text{vol}_{2m}(\mathcal{Q}(C_m) \setminus \mathcal{P}(C_m))^{1/2m} \sim \left(\frac{2^{m-2}}{\sqrt{4m\pi} \left(\frac{2m}{e}\right)^{2m}} \right)^{1/2m} \sim \frac{e}{m\sqrt{2}},$$

and (12) follows. \square

We note that because P_m is a forest, $\mathcal{P}(P_m) = \mathcal{Q}(P_m)$. While of course C_m is not a forest, and so $\mathcal{P}(C_m) \neq \mathcal{Q}(C_m)$. One way this different behavior manifests itself is in the explosion of the number of facets for $\mathcal{P}(C_m)$. But in some sense the graphs P_m and C_m do not look very different, and we see this echoed in the facts that: (i) the asymptotic behavior of their volumes is identical (compare (10) and (11)), and (ii) $\mathcal{Q}(P_m) \setminus \mathcal{P}(P_m) = \emptyset$ while $\text{vol}_{2m}(\mathcal{Q}(C_m) \setminus \mathcal{P}(C_m))^{1/2m}$ decays like $e/m\sqrt{2}$, so it is nearly zero.

Next, for a natural number p , let C_3^p denote a graph that is the disjoint union of p copies of the triangle C_3 . For m divisible by 3, we wish to compare the behaviors of $\mathcal{Q}(C_3^{m/3})$ and $\mathcal{Q}(C_3^{m/3}) \setminus \mathcal{P}(C_3^{m/3})$ with those of $\mathcal{Q}(C_m)$ and $\mathcal{Q}(C_m) \setminus \mathcal{P}(C_m)$.

Corollary 21.

$$\text{vol}_{2m}(\mathcal{Q}(C_3^{m/3}))^{1/2m} = \left(\frac{1}{120} \right)^{1/6} \approx 0.450267 \quad (13)$$

and

$$\text{vol}_{2m}(\mathcal{Q}(C_3^{m/3}) \setminus \mathcal{P}(C_3^{m/3}))^{1/2m} = \left(\frac{1}{360} \right)^{1/6} \approx 0.374929. \quad (14)$$

Proof. Easily follows from Theorem 13, Theorem 17 and Lemma 1. \square

It is very interesting to see, comparing (11) with (13), that $\text{vol}_{2m}(\mathcal{Q}(C_m))$ has a remarkably similar behavior to $\text{vol}_{2m}(\mathcal{Q}(C_3^{m/3}))$, while, comparing (12) with (11) and (14), $\text{vol}_{2m}(\mathcal{Q}(C_m) \setminus \mathcal{P}(C_m))$ is quite small compared to both $\text{vol}_{2m}(\mathcal{Q}(C_m))$ and to $\text{vol}_{2m}(\mathcal{Q}(C_3^{m/3}) \setminus \mathcal{P}(C_3^{m/3}))$. That is, for one long

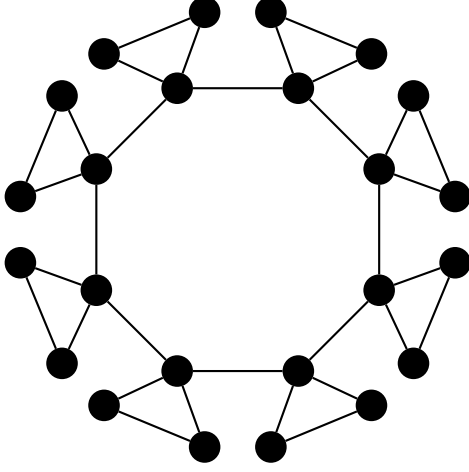


Figure 3: 8-necklace

cycle C_m , the polytope \mathcal{P} , which needs $4m + 2^{m-1}$ facets to describe, is well approximated by the polytope \mathcal{Q} , which needs only $4m$ facets to describe. On the other hand, for a collection of $m/3$ triangles (which has the same number of edges as C_m), the polytope \mathcal{P} only needs $16m/3$ facets to describe and in this case it is not well approximated by \mathcal{Q} , which still needs only $4m$ facets to describe.

7. Computational experiments

In this section, we report on computational experiments designed to see if the behavior of the cycles persists for a family of more complex graphs. Toward this end, for $n \geq 3$, we define the n -necklace N_n to be a cycle C_n with a triangle, i.e. C_3 , “hanging” from each vertex of C_n . In Figure 3, we depict N_8 .

Our goal was to compare the volumes of various relaxations of $\mathcal{P}(N_n)$. We compared them via their d -th roots, following the spirit of §6. The polytope $\mathcal{R}(N_n)$ is the part of the basic relaxation $\mathcal{Q}(N_n)$ that satisfies the 2^{n-1} odd cycle inequalities associated with the big cycle C_n . The polytope $\mathcal{T}(N_n)$ is the part of the basic relaxation $\mathcal{Q}(N_n)$ that satisfies the four odd cycle inequalities associated with each triangle C_3 . Finally, we have the usual boolean quadric polytope $\mathcal{P}(N_n)$ of N_n , which we note is the intersection of $\mathcal{R}(N_n)$ and $\mathcal{T}(N_n)$.

n	\mathcal{Q}	\mathcal{R}	\mathcal{T}	\mathcal{P}
4	0.573963	0.425950	0.396662	0.399680
5	0.573963	0.426805	0.400130	0.399400
6	0.573963	0.426061	0.399665	0.399436
7	0.573963	0.428294	0.400695	0.400313
8	0.573963	0.426034	0.399421	0.400937
9	0.573963	0.425517	0.399619	0.400723
10	0.573963	0.426842	0.401514	0.400903
11	*	0.426218	0.400749	0.400700
12	*	0.426597	0.400482	0.400965

Table 1: Comparison of relaxations of the BQP for n -necklaces

Using **LEcount** (see [21] and [22]), we exactly calculated $(\text{vol}_d(\mathcal{Q}(N_n)))^{1/d}$, for $n = 4, \dots, 10$. These numbers appear in the second column of Table 1. Note that the polytope $\mathcal{Q}(N_n)$ lives in dimension $d = 7n$. We stopped after $n = 10$, due to memory issues. But we can easily observe that to 6 decimal places, we have a clear picture of the limiting constant.

For $n = 4, \dots, 12$, we approximated the volumes of several related polytopes, using the **Matlab** software [12]. These numbers appear in the remaining columns of Table 1. Note that in the **Matlab** software, we varied the precision ε depending on the dimension, so as to approximate the d -th roots to a precision of roughly $\delta = 0.0001$. So we set $\varepsilon := (1 + \delta)^d - 1$.

Generally, $\mathcal{T}(N_n)$ is a very light refinement of $\mathcal{Q}(N_n)$, while $\mathcal{R}(N_n)$ is a rather heavy refinement. For example, for $n = 12$, as compared to $\mathcal{Q}(N_{12})$, $\mathcal{R}(N_{12})$ uses 2048 extra inequalities, while $\mathcal{T}(N_{12})$ uses only 48 extra inequalities, and hence $\mathcal{P}(N_{12})$ uses 2096 extra inequalities. What we can easily see is that we almost completely capture $\mathcal{P}(N_n)$ with the very light $\mathcal{T}(N_n)$. Furthermore, the very heavy $\mathcal{R}(N_n)$ leaves a significant gap to $\mathcal{P}(N_n)$. In summary, the messages of Corollaries 20 and 21 extend to more complicated situations: *odd cycle inequalities seem to be more important for short cycles than long cycles*. We note that this echoes the message of [16].

8. Future work

A very challenging open problem is to get a polynomial-time algorithm for calculating $\text{vol}(\mathcal{P}(G))$ when G is a series-parallel graph (briefly, the class of graphs having no K_4 graph minor). In this case, $\mathcal{P}(G)$ is completely described by F1-F4. Even for the subclass of outerplanar graphs (briefly, the class of graphs having no K_4 nor $K_{2,3}$ graph minor), this is already very challenging because the number of cycles in such a graph can be exponential (in fact $> \Omega(1.5^n)$, see [13]). A more manageable problem would be to restrict our attention to the further subclass of cactus forests, i.e. graphs where each edge is in no more than one cycle — they can also be understood as the class of graphs having no diamond (i.e., K_4 minus an edge) graph minor. The necklaces N_n (see §7) are cactus graphs. Cactus graphs occur in a wide variety of applications, e.g., location theory, communication networks, and stability analysis (see [29] and the references therein). Cactus graphs can be recognized in linear time, via a depth-first search approach (see [29] and [38]), and of course the number of cycles in such a graph is at most $n/3$. We do note that we can apply Lemma 15, which tells us that odd cycle inequalities from the same odd cycle cut off disjoint parts of $\mathcal{Q}(G)$. But it already seems to be quite a challenging problem to characterize the volume of $\mathcal{Q}(G)$ cut off by a single odd cycle inequality (seeking to generalize Lemma 16).

Finally, it is interesting to note that while odd cycle inequalities can be separated in $\mathcal{O}(n^3)$ time (see [4]), we can actually devise a very simple $\mathcal{O}(n^2)$ approach for cactus graphs. First, we enumerate the cycles of the cactus graph. Then for each cycle, we simply have to check, for a given fixed (x, y) if there is an A for which F4 is violated. This simplest way to see how to proceed involves using the affine equivalence of $\mathcal{P}(K_n)$ with the cut polytope of K_{n+1} (see [14]). For the cut polytope (which has variables indexed only by edges), the odd cycle inequalities can be written in the form:

$$\sum_{e \in F} (1 - z_e) + \sum_{e \in C \setminus F} z_e \geq 1, \quad (15)$$

where C is a cycle and $F \subset C$ has odd cardinality. Using an idea in [11, proof of Proposition 1.7], we can let

$$F' := \{e \in C : z_e > 1/2\}.$$

Then z can violate (15) only if $|F \Delta F'| \leq 1$. So either F' has odd cardinality and we need only check (15) for $F = F'$, or F has even cardinality, and we need only check (15) for the at most $|C|$ (odd) sets F satisfying $|F \Delta F'| = 1$.

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